ON THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS AND TIME LAG

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Considered is a system of linear differential equations with periodic coefficients and with after-effect (with time lag). The study of this type of systems is of interest in particular to persons working with automatic control systems.

The investigation is based on methods for treating equations with time lag in the function space $C_{-\tau 0}$ of continuous functions proposed by Krasovskii [1,2].

It is shown that the spectrum of the operator of the monodrome $U(\omega, t_0)$ does not depend on t_0 and determines the asymptotic stability or instability of the motion $x = 0$. A conjugate system of differential equations with time advance and with periodic coefficients is constructed. An explicit expression is given for the first integrals of the considered system (1.1) by means of the solution of the conjugate system. An explanation is given for the connection between the spectra of the operators of the monodrome of the original and conjugate systems; an analytic form of the characteristic vectors and particular solutions of these systems is obtained, which can be continued over the entire time-axis from $-\infty$ to $+\infty$.

It is shown that in the space of continuous functions $C_{-\tau 0}$, in which the solutions of the system (1.1) are considered, there can be found a finite-dimensional basis, periodic in t, on which the motion of the system (1.1) is described by a system of ordinary differential equations

with constant coefficients. In the complementary subspace, the norm of every solution decreases as an exponential function with sufficiently high exponent because the spectral radius of the operator of the monodrome can be made as small as one pleases. The last circumstance has application in the theory of the stability of oscillations, as well as in problems on optimal control in systems with delay.

1. Let us consider a system of differential equations with delay of the type

$$\frac{dx_s(t)}{dt} = F_s(t, x_1(t+\vartheta), \ldots, x_n(t+\vartheta)) \tag{1.1}$$

Here

$$F_s(t, x_1(\vartheta), \ldots, x_n(\vartheta)) = \sum_{j=1}^{n} \sum_{\sigma=1}^{k} p_{sj\sigma}(t)\, x_j(-\tau_\sigma) + \sum_{j=1}^{n} \int_{-\tau}^{0} f_{sj}(t, \xi)\, x_j(\xi)\, d\xi$$

The periodic continuous functions $p_{sj\sigma}(t)$ of time are of period ω. The functions $f_{sj}(t, \xi)$ are continuous in t and in the region $-\tau \leqslant \xi \leqslant 0$, $-\infty < t < +\infty$; they are periodic in t of period ω; τ_σ is the delay of the system.

Let us denote by $x(\varphi(\vartheta), t_0, t)$ the solution of the system (1.1) with the initial function $\varphi(\vartheta) = \{\varphi_s(\vartheta), s = 1, \ldots, n; -\tau \leqslant \vartheta \leqslant 0\}$.

A segment of the trajectory $x(\varphi(\vartheta), t_0, t+\vartheta)$ will be considered to be an element of the solution of the system (1.1). Thus, to the system of equations (1.1) in the function space $C_{-\tau 0}$ of continuous functions on the interval $(-\tau, 0)$ with norm $\| x(\vartheta) \|_{-\tau 0} = \sup(|x_1(\vartheta)|, \ldots, |x_n(\vartheta)|, (-\tau \leqslant \vartheta \leqslant 0)$ there will correspond a system of "ordinary" differential equations with an operator type right-hand side

$$\frac{dx_t(\vartheta)}{dt} = P(t)\, x_t(\vartheta) \qquad (\Delta t > 0) \tag{1.2}$$

where $x_t(\vartheta) = x(t+\vartheta) = \{x_s(t+\vartheta), s = 1, \ldots, n\}$, while the operator $P(t)$ is defined in the following way

$$P(t)\, x(\vartheta) = \tag{1.3}$$
$$= \left\{\frac{dx_k(\vartheta)}{d\vartheta} \text{ when } \tau \leqslant \vartheta < 0, \quad F_k(t, x_1(\vartheta), \ldots, x_n(\vartheta)) \text{ when } \vartheta = 0,\ k = 1, \ldots, n\right\}$$

For a fixed $t > t_0$, an element of the solution $x_t(\vartheta) = x(\varphi(\vartheta), t_0, t+\vartheta)$ can be considered to be an image of the element $\varphi(\vartheta) \in C_{-\tau 0}$ under some mapping

$$x_t(\vartheta) = T(t, t_0)\, \varphi(\vartheta) \qquad (t \geqslant t_0) \tag{1.4}$$

with the operator $T(t, t_0)$; $T(t_0, t_0) = J$ is the identity operator

$$Jx(\vartheta) \equiv x(\vartheta), \quad T(t, t_0)\varphi(\vartheta) = x(\varphi(\vartheta), t_0, t+\vartheta)$$

We shall note some basic properties of the operator $T(t, t_0)$.

1. The operator $T(t, t_0)$ is linear.

2. The operator $T(t, t_0)$ has the property of a semi-group. For every t and t_1 it is true that

$$T(t + t_1, t_0) = T(t_1 + t, t)\, T(t, t_0) \qquad (t_1 > 0,\ t > t_0) \tag{1.5}$$

3. The operator $T(t, t_0)$ satisfies the condition

$$T(t + \omega, t_0) = T(t, t_0)\, T(t_0 + \omega, t_0) \qquad (t > t_0) \tag{1.6}$$

Since the system (1.2) depends in a periodic way on time t, it follows that

$$x_{t+\omega}(\vartheta) = x(\varphi(\vartheta), t_0, t + \omega + \vartheta)$$

is a solution of the system (1.2), and, when $\vartheta = 0$, also of the system (1.1).

But then one can find an element $\varphi^*(\vartheta)$ of the space $C_{-\tau 0}$ such that $x_{t+\omega} = x(\varphi^*(\vartheta), t_0, t+\vartheta)$, $x(\varphi(\vartheta), t_0, t_0 + \omega + \vartheta) = \varphi^*(\vartheta)$. We thus have

$$\varphi^*(\vartheta) = T(t_0 + \omega, t_0)\,\varphi(\vartheta), \qquad x_{t+\omega} = T(t, t_0)\,\varphi^*(\vartheta)$$
$$x_{t+\omega}(\vartheta) = T(t + \omega, t_0)\,\varphi(\vartheta)$$

This implies (1.6).

4. Setting $t_1 = \omega$ in (1.5) and taking into consideration (1.6), we obtain

$$T(t, t_0)\, T(t_0 + \omega, t_0) = T(t + \omega, t)\, T(t, t_0) \tag{1.7}$$

In view of (1.5) and (1.7) we have also

$$x_{t+\omega}(\vartheta) = T(t+\omega, t)\, x_t(\vartheta) = T(t+\omega, t)\, T(t, t_0)\varphi(\vartheta) =$$
$$= T(t, t_0)\, T(t_0 + \omega, t_0)\varphi(\vartheta) = T(t, t_0)\, T(t_0 + \omega, t_0)\, T^{-1}(t, t_0)\, x_t(\vartheta)$$
$$T^{-1}(t, t_0)\, x_t(\vartheta) = \varphi(\vartheta)$$

Whence,

$$x_{t+\omega}(\vartheta) = T(t, t_0)\, T(t_0 + \omega, t_0)\, T^{-1}(t, t_0)\, x_t(\vartheta) \tag{1.8}$$

5. Let $t = n\omega + t^*$, $t_0 \leqslant t^* \leqslant t_0 + \omega$, where n is a positive integer. Applying formula (1.6) n times, we obtain

$$T(t, t_0) = T(t^*, t_0)\, T^n(t_0 + \omega, t_0) \tag{1.9}$$

The operator $T(t_0 + \omega, t_0)$ will play an important role in what follows.* Below we shall denote this operator by the symbol $U(\omega, t_0)$.

2. The operator $U(\omega, t_0) = T(t_0 + \omega, t_0)$ is completely continuous on the linear normed space $C_{-\tau 0}$ of continuous functions because it is bounded in view of (27.12) [1], and it transforms continuous functions $x(\vartheta) \in C_{-\tau 0}$ into uniformly continuous functions ([1], page 226, Vol.25, No.1). Let us consider the equation

$$(U(\omega, t_0) - \rho J)\, x(\vartheta) = 0 \tag{2.1}$$

Here J is the identity operator, ρ is a complex number, and $x(\vartheta) \in C_{-\tau 0}$. Since the operator $U(\omega, t_0)$ is completely continuous, equation (2.1) may have non-trivial solutions for a denumerable set of values of ρ_j. These values are called characteristic numbers of the operator $U(\omega, t_0)$. For each ρ_j equation (2.1) has a finite number n_j of linearly independent characteristic vectors $x^{(j)}(\vartheta)$ of the operator $U(\omega, t_0)$. There exists a number n_0, which is independent of the number j, such that $n_j \leqslant n_0$. The characteristic numbers ρ_j $(j = 1, 2, \ldots)$, and the point $\rho = 0$ constitute the spectrum of the operator $U(\omega, t_0)$. In the region $|\rho| \geqslant r$ (where r is an arbitrary positive number) there exist only a finite number of characteristic numbers of the operator $U(\omega, t_0)$ (see [4]).

Theorem 2.1. The spectrum $\{\rho_j\}$ of the operator $U(\omega, t_0)$ does not depend on t_0. The characteristic vectors $x_t^{(j)}(\vartheta)$, $x_{t_0}^{(j)}(\vartheta)$ of the operators $U(\omega, t)$ and $U(\omega, t_0)$ that correspond to the characteristic number ρ_j are connected by the relations

$$x_t^{(j)}(\vartheta) = T(t, t_0)\, x_{t_0}^{(j)}(\vartheta), \quad x_{t_0}^{(j)}(\vartheta) = T^{-1}(t, t_0)\, x_t^{(j)}(\vartheta) \tag{2.2}$$

and $x_t^{(j)}(\vartheta)$ has the form

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* This operator was used by Iu.M. Repin for the investigation of equations with constant coefficients [7].

$$x_t^{(j)}(\vartheta) = \rho_j^{\frac{t+\vartheta}{\omega}} u_j(t+\vartheta) \tag{2.3}$$

where $u_j(t+\vartheta)$ is a periodic vector-function of period ω in t; the function $x_t^{(j)}(\vartheta)$ satisfies the system (1.2) not only when $t > t_0$ but also when $t < t_0$.

Proof. Let $x_{t_0}^{(j)}(\vartheta)$ be a characteristic vector of the operator $U(\omega, t_0)$ (t_0 is arbitrary) which corresponds to the number ρ_j. The following identity is valid

$$(U(\omega, t_0) - \rho_j J)\, x_{t_0}^{(j)}(\vartheta) \equiv 0 \tag{2.4}$$

Applying the operator $t(t, t_0)$, $t > t_0$, to the left and right sides of the identity (2.4), and taking into account (1.7), we obtain the identity

$$U(\omega, t)\, T(t, t_0)\, x_{t_0}^{(j)}(\vartheta) \equiv \rho_j T(t, t_0)\, x_{t_0}^{(j)}(\vartheta) \tag{2.5}$$

From (2.5) it follows that ρ_j will be a characteristic number, and $x_t^{(j)}(\vartheta) = T(t, t_0) x_{t_0}^{(j)}(\vartheta)$ a characteristic vector, also of the operator $U(\omega, t)$, $t > t_0$. The function

$$x_t^{(j)}(\vartheta) = T(t, t_0)\, x_{t_0}^{(j)}(\vartheta)$$

with $t > t_0$, will also be a solution of the system (1.2). Taking into account (1.7) and (1.8) we obtain from (2.5)

$$x_{t+\omega}^{(j)}(\vartheta) = \rho_j\, x_t^{(j)}(\vartheta) \qquad (t > t_0) \tag{2.6}$$

Formula (2.6) implies formula (2.3) when $t > t_0$. It is, however, easy to notice that $x_t^{(j)}(\vartheta)$, determined by the formula (2.3) when $t < t_0$, also satisfies the system (2.1). Assuming that $t_1 = t + l\omega > t_0$ (l an integer) we find that $x_{t_1}^{(j)}(\vartheta)$ satisfies the system (1.2), where t is replaced by t_1. Taking into account the periodicity of $P(t)$ and $x_{t_1}^{(j)}(\vartheta) = x_{t+l\omega}^{(j)}(\vartheta) = \rho_j^{l\omega} x_t^{(j)}(\vartheta)$, one can verify that the function $x_t^{(j)}(\vartheta)$, determined by formula (2.3), satisfies equation (1.2) when $t < t_0$.

Let us suppose that $x_{t_1}^*(\vartheta)$ is a characteristic vector, ρ^* is a characteristic number of the operator $U(\omega, t_1)$, while $x_t(\vartheta)$ is a solution of the system (1.2) with the initial function $x_{t_1}(\vartheta)$ at the time t_1. This solution can be continued over the entire real axis t. Hence one can find a function $x_{t_0}^*(\vartheta)$ such that $x_{t_0}^*(\vartheta) = T^{-1}(t, t_0)x_t^*(\vartheta)$. Taking into account the last formula and (1.7) we obtain, from the identity $U(\omega, t_1)x_{t_1}(\vartheta) \equiv \rho^* x_{t_1}(\vartheta)$, the identity $U(\omega, t_0)x_{t_0}^*(\vartheta) \equiv \rho^* x_{t_0}^*(\vartheta)$.

Thus, the spectrum of the operator $U(\omega, t_0)$ is independent of t_0, and the formulas (2.2) and (2.3) are valid.

The system of equations (1.2) has in general a denumerable number of particular solutions defined on the entire axis of time t. Let us assume that $\rho = \rho^*$ is such that the equation

$$(U(\omega, t_0) - \rho^* J)^k x(\vartheta) \equiv 0$$

has a nontrivial solution. Then the system of equations (1.2) will possess a solution of the form

$$\frac{(t+\vartheta)^{k-1}}{(k-1)!} u_1(t+\vartheta) + \frac{(t+\vartheta)^{k-2}}{(k-2)!} u_2(t+\vartheta) + \ldots + u_k(t+\vartheta)$$

where $u_1(t+\vartheta), \ldots, u_k(t+\vartheta)$ are vector-functions of period ω in t.

3. We suppose now that all the characteristic numbers ρ satisfy the condition $|\rho_j| < 1$. It is known [3] that the spectral radius r_u of the operator $U(\omega, t_0)$ is determined by the formula

$$r_u = \lim \| U^n(\omega, t_0) \|^{1/n} \quad \text{on } [-\tau 0] \quad \text{when } n \to \infty$$

Therefore, there exists a number l such that

$$\| U^l(\omega, t_0) \|_{-\tau 0}^{1/l} = q < 1 \quad \text{on } [-\tau, 0]$$

Let $\| T(t, t_0) \| < K$ on the interval $[-\tau, 0]$ when $t_0 \leqslant t \leqslant t_0 + \omega l$, $t = \omega l \sigma + t^*$, $t_0 \leqslant t^* \leqslant t_0 + \omega l$. Then it follows from (1.9) that

$$\| T(t, t_0) \| < \| T(t^*, t_0) \|, \qquad \| U^l(\omega, t_0) \|^\sigma < K q^\sigma \tag{3.1}$$

The following inequality now applies

$$\| T(t, t_0) \|_{-\tau 0} < K e^{-\alpha(t-t_0)} e^{\alpha \omega l} \qquad \left(\alpha = -\frac{\log q}{\omega}\right) \tag{3.2}$$

for all $t > t_0$, $r_u < q < 1$. Every solution of the system (1.1) decreases in norm faster than the exponents in (3.2). The motion $x = 0$ is asymptotically stable. If among the characteristic numbers ρ_j of the operator $U(\omega, t_0)$ there exists one whose modulus is greater than one, then the motion $x = 0$ is unstable. Among the solutions of the system (1.1) there will exist some that increase unboundedly when $t \to \infty$.

4. Along with the system (1.1) we shall consider also a "conjugate" system of differential equations with an advance in time of the form

$$\frac{dy_s(t)}{dt} = -F_s^*(t, y_1(t+\vartheta), \ldots, y_n(t+\vartheta)) \qquad (s = 1, \ldots, n) \quad (4.1)$$

Here,

$$F_s^*(t, y_1(\vartheta), \ldots, y_n(\vartheta)) = \sum_{j=1}^{n} \sum_{\sigma=1}^{k} p_{js\sigma}(t+\tau) y_j(\tau_\sigma) + \sum_{j=1}^{n} \int_{-\tau}^{0} f_{js}(t-\xi, \xi) y_j(-\xi) d\xi$$

The functions $p_{sj\sigma}(t)$, $f_{js}(t, \xi)$ are the same as in the system (1.1). The system of equations (4.1) plays the part of a conjugate system in the theory of linear differential equations with periodic coefficients.

Let us denote by $y(y_0(\vartheta), t_0, t)$ the solution of the system (4.1), when $t < t_0$, with the initial function $y_0(\vartheta)$, $\tau \geqslant \vartheta \geqslant 0$ for $t = t_0$. In the nature of an element of a solution we shall consider a section (segment) of the trajectory on the interval $[t+\tau, t]$ $y(y_0(\vartheta), t_0, t+\vartheta)$. To the system of equations (4.1) in the function space $C_{-\tau 0}$ of continuous functions $y(\vartheta)$ on the interval $\tau \geqslant \vartheta \geqslant 0$ with the norm $\| y(\vartheta) \|$ sup $(|y_1(\vartheta)|, \ldots, |y_n(\vartheta)|, \tau \geqslant \vartheta \geqslant 0)$, there will correspond now a system of "ordinary" differential equations with an operator-type right hand side

$$\frac{dy_t(\vartheta)}{dt} = -P^*(t) y_t(\vartheta) \qquad (\Delta t < 0) \qquad (4.2)$$

where

$$y_t(\vartheta) = y(t+\vartheta) = \{y_s(t+\vartheta), \ \ \tau \geqslant \vartheta \geqslant 0, s = 1, \ldots, n\}$$

The operator $P^*(t)$ is defined in the following way

$$-P^*(t) y(\vartheta) =$$

$$= \left\{ \frac{dy_k(\vartheta)}{d\vartheta} \text{ when } \tau \geqslant \vartheta > 0, \ -F_k^*(t, y_1(\vartheta), \ldots, y_n(\vartheta)), \ \vartheta = 0, k = 1, \ldots, n \right\}$$

For a fixed $t (t < t_0)$ the element of the solution $y_t(\vartheta) = y(y_0(\vartheta), t_0, t+\vartheta)$ $(\tau \geqslant \vartheta \geqslant 0)$ of the system (4.2) can be considered as an image of the element $y(\vartheta)$ for some mapping

$$y_t(\vartheta) = T^*(t, t_0) y_0(\vartheta) \qquad (t < t_0) \qquad (4.3)$$

with the operator $T^*(t, t_0)$; $T^*(t_0, t_0) = J$, where J is the identity operator; $T^*(t, t_0)y_0(\vartheta) = y(y_0(\vartheta), t_0, t+\vartheta)$, $t < t_0$. Let us form the operator

$$T^*(t_0-\omega, t_0) = U^*(\omega, t_0) \tag{4.4}$$

where ω is the period of the system (1.1). The operator $U^*(\omega, t_0)$ is completely continuous and plays the same role for the system (4.1) as the operator $U(\omega, t_0)$ plays for the system (1.1). The operators $T^*(t, t_0)$ and $U^*(t, t_0)$ have the same properties as the operators T and U mentioned in Sections 1 to 3, except for the direction of decrease of time. This latter fact is obvious since, if one replaces t by $-t$, the system (4.1) goes over into a system with time delay of the type (1.1). In particular, to every characteristic number ρ_j of the operator $U^*(\omega, t_0)$ there corresponds a solution of the system (4.1) which can be continued over the entire real axis of t $(-\infty, +\infty)$.

5. Let us introduce the notation

$$\begin{aligned}(x(\vartheta), y(\vartheta), t) = \sum_{j=1}^{n} x_j(0)\, y_j(0) + \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{s=1}^{k}\int_{0}^{\tau_s} x_l(\xi-\tau_s)\, y_j(\xi)\, p_{jls}(t+\xi)\, d\xi - \\ -\sum_{j=1}^{n}\sum_{l=1}^{n}\int_{-\tau}^{0}\Big[\int_{-\vartheta}^{0} x_l(\xi+\vartheta)\, y_j(\xi)\, f_{jl}(\xi+t, \vartheta)\, d\xi\Big] d\vartheta\end{aligned} \tag{5.1}$$

Here $\quad x(\vartheta) = \{x_s(\vartheta), -\tau \leqslant \vartheta \leqslant 0\}, \qquad y(\vartheta) = \{y_s(\vartheta), \tau \geqslant \vartheta \geqslant 0\}.$

By direct computation we obtain in view of the system (1.2) and (4.2) the identity

$$(P(t)\, x(\vartheta), y(\vartheta), t) + (x(\vartheta), -P^*(t)\, y(\vartheta), t) \equiv -\frac{\partial (x(\vartheta), y(\vartheta), t)}{\partial t} \tag{5.2}$$

From the identity (5.2) it follows that for every particular solution $y_t(\vartheta)$ of the system (4.1), which is extendable in the direction of increasing time $t (t \geqslant t_0)$, the expression

$$(x_t(\vartheta), y_t(\vartheta), t) = C \tag{5.3}$$

will be a first integral of the system (1.1) and (1.2) since for any solution $x_t(\vartheta)$ of the systems (1.1) the expression in (5.3) will be constant. From (5.3) we have for the mentioned solutions $x_t(\vartheta)$ and $y_t(\vartheta)$ the equation

$$(x_{t+\omega}(\vartheta), y_{t+\omega}(\vartheta), t) - (x_t(\vartheta), y_{t+\omega}(\vartheta), t) = (x_t(\vartheta), y_t(\vartheta), t) - (x_t(\vartheta), y_{t+\omega}(\vartheta), t)$$
$$(U(\omega, t)\, x_t(\vartheta),\ \rho^{-1} y_t(\vartheta), t) = (x_t(\vartheta), U^*(\omega, t)\, \rho^{-1} y_t(\vartheta), t)$$

Thus, for any characteristic element $y(\vartheta)$ of the operator $U^*(\omega, t_0)$ and for any $x(\vartheta) \in C_{-T0}$, we have the identity

$$(U(\omega, t)\, x(\vartheta), y(\vartheta), t) = (x(\vartheta), U^*(\omega, t)\, y(\vartheta), t) \tag{5.4}$$

Theorem 5.1. The characteristic numbers $\{\rho_j\}$ and $\{\rho_j^*\}$ of the operators $U(\omega, t_0)$ and $U^*(\omega, t_0)$ are the same ($\rho_j = \rho_j^*$, $j = 1, 2, \ldots$).

Proof. Let ρ^* be a characteristic number of the operator $U^*(\omega, t_0)$. We shall show that this number is also a characteristic number of the operator $U(\omega, t_0)$. For this it is sufficient to show that the equation $(U(\omega, t_0) - \rho^* I)\, x(\vartheta) = 0$ has a nontrivial solution. Let us assume the opposite, that the latter operator does not vanish identically for any $x(\vartheta) \in C_{-T0}$ ($x(\vartheta)$ different from zero). Suppose that to the number ρ^* there corresponds a particular solution of the system (4.1):

$$\rho^{*\frac{t+\vartheta}{\omega}}\, v\,(t + \vartheta)$$

Making use of (5.3) or (5.4), we find that for every $x(\vartheta)$ the following condition holds:

$$((u\,(\omega, t_0) - \rho^* J)\, x\,(\vartheta),\ \rho^{*-(t_0+\vartheta)/\omega}\, v\,(t_0 + \vartheta), t_0) = 0 \tag{5.5}$$

Thus, the operator $(U(\omega, t_0) - \rho^* J)$ has for its range of values a subspace Λ of the space C_{-T0}:

$$(x\,(\vartheta),\ \rho_j^{*-(t_0+\vartheta)/\omega}\, v\,(t_0 + \vartheta), t_0) = 0 \tag{5.6}$$

On this subspace Λ (5.6), we consider the equation

$$(U\,(\omega, t_0) - \rho^* J)\, x\,(\vartheta) = x^*\,(\vartheta) \tag{5.7}$$

where $x^*(\vartheta)$ is an arbitrary element of Λ. In order that the equation (5.7) may be solvable for any $x^*(\vartheta) \in \Lambda$, it is sufficient that the homogeneous equation have a unique solution $x(\vartheta) \equiv 0$ on Λ [4] (Theorem 2, page ...). In view of the hypothesis made earlier, the equation (5.7) has a single solution and it belongs to Λ.

Let us next select an arbitrary element $X(\vartheta) \notin \Lambda$. We consider the element $x(\vartheta) = z(\vartheta) + X(\vartheta)$, where $z(\vartheta) \in \Lambda$ satisfies the equation

$$(U\,(\omega, t_0) - \rho^* J)\, z\,(\vartheta) = -\,(U\,(\omega, t_0) - \rho^* J)\, X\,(\vartheta)$$

The last equation is solvable by hypothesis. But then $z(\vartheta) + X(\vartheta)$ is a characteristic element of the operator $U(\omega, t_0)$ corresponding to the number ρ^*. Therefore, ρ^* is characteristic number of the operator

$U(\omega,\ t_0)$.

In an analogous manner one can show that the converse is also true. The theorem has thus been proved.

Furthermore, one can show that the cardinal numbers of the sets of roots of the equations

$$(U(\omega, t_0) - \rho J)^k x(\vartheta) = 0, \qquad (U^*(\omega, t_0) - \rho J)^k y(\vartheta) = 0 \tag{5.8}$$

are the same if k is any positive integer.

Let us consider the equation

$$(U(\omega, t_0) - \rho J)\, x(\vartheta) = b(\vartheta) \qquad (b(\vartheta) \in C_{-\tau 0}) \tag{5.9}$$

where ρ is a complex number.

If ρ is not a characteristic number of the operator $U(\omega,\ t_0)$, then the equation (5.9) has a unique solution. If ρ is a characteristic number of the operator $U(\omega,\ t_0)$ then it will also be a characteristic number of the operator $U^*(\omega,\ t_0)$. Let $y_1(\vartheta),\ \ldots,\ y_m(\vartheta)$ be independent characteristic elements of the operator $U(\omega,\ t_0)$. Then the necessary and sufficient conditions for the solvability of the equation (5.9) are the following m relations

$$(b(\vartheta), y_j(\vartheta), t_0) = 0 \qquad (j = 1, \ldots, m) \tag{5.10}$$

We will omit the proof of this assertion. Making use of the last stated fact and of (5.4), we obtain the following result. Suppose that the operator $U(\omega,\ t_0)$ has a finite or a denumerable number of characteristic numbers ρ_j. Then one can construct for the operators $U(\omega,\ t_0)$ and $U^*(\omega,\ t_0)$ sequences of root elements $x_j(\vartheta)$, $y_j(\vartheta)$ such that they will satisfy the following conditions.

If $x_j(\vartheta)$ is a characteristic element, then

$$(x_j(\vartheta), y_\sigma(\vartheta), t_0) = \begin{cases} 1, & j = \sigma \\ 0, & j \neq \sigma \end{cases} \tag{5.11}$$

If $x_j(\vartheta),\ x_{j+1}(\vartheta),\ \ldots,\ x_{j+m}(\vartheta)$ constitute a Jordan chain (x_j is a characteristic element, while $x_{j+1},\ \ldots,\ x_{j+m}$ are adjoining elements) for the operator $U(\omega,\ t_0)$, then the corresponding Jordan chain for the conjugate operator $U^*(\omega,\ t_0)$ will be $y_j(\vartheta),\ y_{j+1}(\vartheta),\ \ldots,\ y_{j+m}(\vartheta)$ and the next conditions will hold

$$(x_{j+k}(\vartheta), y_{j+m-\sigma}(\vartheta), t_0) = \begin{cases} 1, & \sigma = k \\ 0, & \rho \neq k \end{cases} \quad \begin{pmatrix} 0 \leqslant \sigma \leqslant m \\ 0 \leqslant k \leqslant m \end{pmatrix} \tag{5.12}$$

$$(x_{j+k}(\vartheta), y_\sigma(\vartheta), t_0) = 0 \qquad (\sigma < j, \sigma > j + m) \tag{5.13}$$

6. Let us construct $N(\varepsilon)$ particular solutions of the systems (1.2) and (4.2) $x_{j+k}(t+\vartheta)$, $y_{j+k}(t+\vartheta)$ which correspond to the $N(\varepsilon)$ characteristic numbers ρ_j satisfying the conditions $|\rho_j| \geqslant \varepsilon$ (ε is any arbitrarily small positive number).

These solutions have the form

$$\begin{aligned} x_{j+k}(t+\vartheta) &= \rho_j^{(t+\vartheta)/\omega}\Phi_{j+k}(t+\vartheta, u(t+\vartheta)) \qquad (-\tau \leqslant \vartheta \leqslant 0) \\ y_{j+k}(t+\vartheta) &= \rho_j^{-(t+\vartheta)/\omega}\Phi_{j+k}(t+\vartheta, v(t+\vartheta)) \qquad (\tau \geqslant \vartheta \geqslant 0) \end{aligned} \quad (k=0,1,\ldots,m) \tag{6.1}$$

Here

$$\Phi_{j+k}(t_1, u(t)) = \frac{t_1^k}{k!} u_j(t) + \ldots + t_1 u_{j+k-1}(t) + u_{j+k}(t) \tag{6.2}$$

The vector-functions $u_j(t), \ldots, u_{j+k}(t), v_j(t), \ldots, v_{j+k}(t)$ have the period ω.

If the adjoint elements do not correspond to the ρ_j, then $k = 0$ in the solutions (6.1); if m of the adjoint elements correspond to the ρ_j, then $k = 0, 1, \ldots, m$ in the formulas (6.1). If the relation (5.11) holds, then we have

$$(\Phi_j(\vartheta, u(t+\vartheta)), \Phi_\sigma(\vartheta, v(t+\vartheta), t) = \begin{cases} 1, & j = \sigma \\ 0, & j \neq \sigma \end{cases} \tag{6.3}$$

If the relations (5.12) and (5.13) are valid, then we obtain

$$(\Phi_{j+k}(\vartheta, u(t+\vartheta)), \Phi_{j+m-\sigma}(\vartheta, v(t+\vartheta), t) = \begin{cases} 1, & \sigma = k \\ 0, & \sigma \neq k \end{cases} \quad (\sigma, k = 0, \ldots, m) \tag{6.4}$$

$$(\Phi_{j+k}(\vartheta, u(t+\vartheta)), \Phi_{j\sigma}(\vartheta, v(t+\vartheta)), t) = 0 \qquad (\sigma < j, \sigma > j+m) \tag{6.5}$$

We shall represent an arbitrary element $x(\vartheta)$ of the space $C_{-\tau 0}$ by means of the formula

$$x(\vartheta) = \sum_{j=1}^{N(\varepsilon)} a_j \Phi_j(\vartheta, u(t+\vartheta)) \rho_j^{(t_0+\vartheta)/\omega} + z_t(\vartheta) \tag{6.6}$$

where

$$a_j = (x(\vartheta), \rho_j^{-(t_0+\vartheta)/\omega} \Phi_j(\vartheta, v(t+\vartheta)), t)$$

if the formula (6.3) holds for the numbers j, and

$$a_{j+k} = (x(\vartheta),\ \rho_j^{-(t_0+\vartheta)/\omega}\Phi_{j+m-k}(\vartheta,\ v(t+\vartheta)),\ t) \qquad (k=0,\ldots,m) \quad (6.7)$$

if the formulas (6.4) and (6.5) apply for the elements numbered from j to $j + m$. Then $z_i(\vartheta)$ in (6.6) satisfies the conditions

$$f_j(z_t(\vartheta)) \equiv (z_t(\vartheta)\,\rho_j^{-\vartheta/\omega}\Phi_j(\vartheta,\ v(t+\vartheta)),\ t) = 0 \qquad (j=1,\ldots,N(\varepsilon)) \ (6.8)$$

It is not difficult to show that the representation (6.6) determines the $a_1, \ldots, a_N$, $z_t(\vartheta)$ in a unique way.

Evaluating the expression $dx_t(\vartheta)/dt - P(t)x_t(\vartheta)$, and assuming that $x(\vartheta)$ has the representation (6.6), we obtain a system of differential equations in the variables $a_1, \ldots, a_N$, $z(\vartheta)$, which corresponds to the system (1.2)

$$\frac{da_j}{dt} = \frac{1}{\omega} a_j \log \rho_j - a_{j+1} \qquad (j=1,\ldots,N(\varepsilon)) \qquad (6.9)$$

$$\frac{dz_t(\vartheta)}{dt} = P(t)\,z_t(\vartheta), \qquad f_j[z_t(\vartheta)] = 0 \qquad (j=1,\ldots,N(\varepsilon)) \qquad (6.10)$$

In equation (6.9) the last term a_{j+1} will be absent if there are no adjoint elements that correspond to the element j; if to the jth element there correspond m adjoint ones, then the term a_{j+1} will appear in all equations with numbers from j to $j + m - 1$, and it will not enter into the equation with the number $j + m$.

It is not difficult to show that if $z_t(\vartheta)$ satisfies the conditions (6.8), then the element $dz_t(\vartheta)/dt - P(t)z_t(\vartheta)$ will also belong to the subspace (6.8).

If one constructs the operator $U(\omega, t_0)$ on the subspace (6.8) for the equation (6.10), then one can easily convince oneself that its spectrum will contain all characteristic numbers ρ_j of the operator $U(\omega, t_0)$ on $C_{-\tau 0}$, except the first $N(\varepsilon)\rho_j$ $(j = 1, \ldots, N(\varepsilon))$, which satisfy the condition $|\rho_j| \geqslant \varepsilon$. Therefore, the norm of every solution $z_t(\vartheta)$ of the equation (6.10) will decrease faster than the exponents $L \exp t/\omega \log \varepsilon$, where $\varepsilon < 1$, while L is some positive number.

Thus, we have arrived at the following result. In the space $C_{-\tau 0}$ one can find a periodically shifting N-dimensional basis $\{\Phi(\vartheta,\ u(t+\vartheta))\ \rho_j^{(t_0+\vartheta)/\omega}\}$ such that in the N-dimensional space determined by this basis, the system (1.2) will describe an N-dimensional system of ordinary differential equations with constant coefficients (6.9).

Hereby, the indicated motion of the system (1.2), having the form of that treated by Floquet, will be an asymptotical solution, when $t \to \infty$, of the system (1.2) in the entire space $C_{-\tau 0}$. The vanishing complement $z_t(\vartheta)$ of the finite-dimensional motion $\{a_j\}$ to the total motion $x_t(\vartheta)$ will remain for all $t > t_0$ in the linear subspace determined by the condition (6.8).

Thus the theory of Floquet for systems with time lag will always be valid in the asymptotic sense.

We note that the system of the root elements $x_\sigma(\vartheta)$, generally speaking, does not possess the property of completeness. In the work [5], Zamerkin gave an example in which an equation with periodic coefficients and with time lag has only one root element. This equation has the form

$$\frac{dx(t)}{dt} = p(t)\,x(t-\tau), \qquad \int_{-\tau}^{0} p(\xi),\,d\xi = 0, \qquad p(t+\tau) = p(t)$$

It must, however, be recognized that the presence of a denumerable system of root elements for an equation of the type (1.1) is rather the rule than the exception. One can convince oneself of this through the consideration of systems with constant coefficients, or of systems with periodic coefficients which are near to constant ones.

BIBLIOGRAPHY

1. Krasovskii, N.N., *Nekotorye zadachi teorii ustoichivosti dvizheniia (Some problems of the theory of stability of motion)*. Fizmatgiz, 1959.

2. Krasovskii, N.N., Ob ustoichivosti kvazilineinykh sistem s posledeistviem (On the stability of quasilinear systems with aftereffect). *Dokl. Akad. Nauk SSSR*, Vol. 119, No. 3, 1958.

3. Gel'fand, I.M., Normirovannye kol'tsa (Normed rings). *Matem. sb.*, 9, pp. 3-24, 1941.

4. Kantorovich, A.V. and Akilov, G.P., *Funktsional'nyi analiz v normirovannykh prostranstvakh (Functional analysis in normed spaces)*. Fizmatgiz, 1959.

5. Zverkin, A.M., K teorii lineinykh differentsial'nykh uravnenii s zapazdyvaiushchim argumentom (On the theory of linear differential equations with lagging argument). *Dokl. Akad. Nauk SSSR*, Vol.128, pp. 882-885, 1959.

6. Khalanai, A., Periodicheskie resheniia lineinykh sistem s zapazdyvaniem (Periodic solutions of linear systems with retardation). *Revue de mathématiques pures et appliquées*, Vol. 6, No. 1, 1961.

7. Repin, Iu.M., Ob usloviiakh ustoichivosti sistem lineinykh differentsial'nykh uravnenii pri liubykh zapazdyvaniiakh (On the conditions of stability of systems of linear differential equations with arbitrary delay). *Uch. zap. Ural'sk. un-ta*, Vol. 23, No. 2, 1960.

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